On Geometric Engineering of $N = 1$ ADE Quiver Models [1]

Adil Belhaj[2]

Instituto de Física Teórica, C-XVI, Universidad Autónoma de Madrid
E-28049-Madrid, Spain

Abstract

In this talk, we discuss four-dimensional $N = 1$ affine ADE quiver gauge models using the geometric engineering method in M-theory on G_2 manifolds with K3 fibrations.

---

[1]Talk presented at the Workshop on Quantum Field Theory, Geometry and Non Perturbative Physics, Rabat, July 28th-29th, 2003.

[2]adil.belhaj@uam.es

1 Introduction

A well-known way to get supersymmetric quantum field theory (QFT) from superstrings, M- or F-theory, is to consider compactifications on a singular manifold X with K3 fibration over a base space B. This method is called geometric engineering [1, 2, 3, 4, 5, 6, 7, 8]. In this way, the gauge group G and matter content of the QFT are defined by the singularities of the fiber and the non-trivial geometry of the base space, respectively. The gauge coupling g is proportional to the inverse of the square root of the volume of the base: $V(B) = g^{-2}$. The complete set of physical parameters of the QFT is related to the geometric moduli space of the internal manifold. For instance, several exact results for the Coulomb branch of $N = 2$ four-dimensional QFT embedded in type IIA superstring theory are obtained naturally by the toric geometry realization and local mirror symmetry of Calabi-Yau threefolds. The latter are realized as a K3 fibration with ADE singularity over a projective $\mathbf{P^1}$ complex curve or a collection of intersecting $\mathbf{P^1}$ curves. The corresponding $N = 2$ QFT$_4$ are represented by quiver diagrams similar to Dynkin diagrams. One should distinguish three cases: ordinary, affine and indefinite Lie algebras [6, 7, 8].

Quite recently, four-dimensional gauge theories preserving only four supercharges have attracted a lot of attention. Work has been done using either intersecting D-brane physics [9, 10, 11, 12, 13, 14], geometric transition in type II superstrings on the conifold [15, 16, 18, 19, 20, 21], or M-theory on G_2 manifolds [22, 23, 24].

The aim of this talk is to discuss $N = 1$ ADE quiver gauge models from M-theory compactification on a seven-dimensional manifold with G_2 holonomy group. The manifold is realized explicitly as K3 fibrations over a three-dimensional base space with Dynkin geometries. First we review the geometric engineering of $N = 2$ QFT$_4$ in type IIA superstring theory on Calabai-Yau threefolds. We then extend this to geometric engineering of $N = 1$ QFT$_4$ in M-theory on G_2 manifolds. These manifolds are given by K3 fibration over three-dimensional spaces with affine ADE Dynkin geometries. Relating our construction to the related D6-brane scenario described using the method of (p, q) webs used in type II superstrings on Calabi-Yau threefolds, we discuss the gauge group and matter content of our $N = 1$ QFT$_4$.

2 Generalities on K3 surfaces with ADE singularities

Since the geometric engineering method is based on the compactification on manifolds with K3 fibration, we first briefly review some basic facts about ADE geometries of this fibration useful for studying supersymmetric gauge theories embedded in superstrings and M-theory compactifications on G_2 manifolds.

Roughly speaking, ADE geometries are singular four-dimensional manifolds asymptotic to R^4/Γ, where Γ is a discrete subgroup of $SU(2)$. The resolution of the singularities of these surfaces is nicely described using the Lie algebra structures. The well known singularities of these spaces are as follows:

1. Ordinary singularities classified by the finite ADE Lie algebras.

2. Affine singularities classified by the affine ADE Kac-Moody algebras.

More general singularities classified by indefinte Lie algebra have been studied in [8]. The latter are at the basis of the derivation of the new four-dimensional $N = 2$ superconformal field theories.

For simplicity, let us focus our attention on reviewing briefly the main lines of the finite ADE geometries. Extended singularities can be found in [6, 7, 8].

In C^3 parametrised by the local coordinates x, y, z, ordinary ADE geometries are described by the following complex surfaces

$$
\begin{aligned}
f(A_n): & \quad xy + z^n = 0 \\
f(D_n): & \quad x^2 + y^2 z + z^{n-1} = 0 \\
f(E_6): & \quad x^2 + y^3 + z^4 = 0 \\
f(E_7): & \quad x^2 + y^3 + yz^3 = 0 \\
f(E_8): & \quad x^2 + y^3 + z^5 = 0.
\end{aligned}
\tag{2.1}
$$

The C^3 origin, $x = y = z = 0$, is a singularity since $f = df = 0$. These geometries can be 'desingularized' by deforming the complex structure of the surface or varying its Kahler structure. The two deformations are equivalent due to the self-mirror property of the ADE geometries . This is why we shall restrict ourselves hereafter to giving only the Kahler deformation. The latter consists in blowing up the singularity by a collection of intersecting complex curves. This means that we replace the singular point $(x, y, z) = (0, 0, 0)$ by a set of intersecting complex curves $\mathbf{P^1}$ (two-cycles). The nature of the set of intersecting $\mathbf{P^1}$ curves depends on the type of the singular surface one

3

is considering. The smoothed ADE surfaces share several features with the ADE Dynkin diagrams. In particular, the intersection matrix of the complex curves used in the resolution of the ADE singularities is, up to some details, minus the ADE Cartan matrix K_{ij}. This link leads to a nice correspondence between the ADE roots α_i and two-cycles involved in the deformation of the ADE singularities. More specifically, to each simple root α_i, we associate a single $(\mathbf{P^1})_i$. This nice connection between the geometry of ADE surfaces and Lie algebra turns out to be at the basis of important developments in string theory. In particular, the abovementioned link has been successfully used in:

(i) The geometric engineering of the $N = 2$ supersymmetric four-dimensional quantum field theory embedded in type II superstrings on Calab-Yau in the presence of D-branes [6, 7, 8].

(ii) Geometric engineering of $N = 1$ quiver theories using conifold geometric transitions [15, 16, 18, 19, 20, 21].

3   Geometric engineering of $N = 2$ QFT in four dimensions

In this section we review the geometric engineering method of $N = 2$ models embedded in type IIA superstring in four dimensions. Then we extend this method to engineering $N = 1$ models in the context of M-theory on G_2 manifolds.

3.1 $SU(2)$ Yang-Mills in six dimensions

The main steps in getting $N = 2$ four-dimensional QFT from type IIA superstrings on Calabi-Yau threefolds is to start first with the propagation of type IIA superstrings on K3 surfaces in the presence of D2-branes wrapping on two-cycles. Then we compactify the resulting model down to four dimensions. To illustrate the method, suppose that K3 has an $su(2)$ singularity. In the vicinity of the $su(2)$ singularity, the fiber K3 may be described by the following equation

$$xy = z^2,$$

where x, y and z are complex variables. The deformation of this singularity consists in replacing the singular point $x = y = z = 0$ by a $\mathbf{P^1}$ curve parameterized by a new variable x' defined as $x' = \frac{x}{z}$. In the new local coordinates (x', y, z), the equation of the A_1 singularity may be written as:

$$x'y = z \tag{3.1}$$

which is not singular. The next step is to consider the propagation of type IIA superstrings in this background in the presence of a D2-brane wrapping around the blown-up $\mathbf{P^1}$ curve (real two-sphere) parametrized by x'. This gives two $W_\mu^\pm$ vector particles, one for each of the two possible orientations for wrapping. These particles have mass proportional to the volume of the blown-up real two-sphere. $W_\mu^\pm$ are charged under the $U(1)$ field Z_0^μ obtained by decomposing the type IIA superstring three-form in terms of the harmonic form on the two-sphere. In the limit where the blown-up two-sphere x' shrinks, we get three masslesss vector particles $W_\mu^\pm$ and Z_0^μ which form an $SU(2)$ adjoint representation. We thus obtain an $N = 2$ $SU(2)$ gauge symmetry in six dimensions.

3.2 $N = 2$ models in four dimensions

A further compactification on a base B_2, that is on a real two-sphere, gives $N = 2$ pure $SU(2)$ Yang-Mills in four dimensions. This geometric $SU(2)$ gauge theory analysis can be easily extended to all simply-laced ADE gauge groups. To incorporate matter, one should consider a non trivial geometry on the base B_2 of the Calabi-Yau threefolds. For example, if we have a two-dimensional locus with $SU(n)$ singularity and another locus with $SU(m)$ singularity and they meet at a point, the mixed wrapped two-cycles will now lead to (n, m) $N = 2$ bi-fundamental matter of the $SU(n) \times SU(m)$ gauge symmetry in four dimensions. Geometricaly, this means that the base geomerty of the Calabi-Yau threefolds is given by two intersecting $\mathbf{P^1}$ curves, according to A_2 finite Dynking diagram, whose volumes V_1 and V_2 define the gauge coupling constants g_1 and g_2 of the $SU(n)$ and $SU(m)$ gauge symmetries, respectively. Fundamental matter is given by taking the limit V_2 to infinity, or equivalently $g_2 = 0$, so that the $SU(m)$ group becomes a flavor symmetry. Geometric engineering of the $N = 2$ four-dimensional QFT shows moreover that the analysis we have been describing recovers naturally some remarkable features which follows from the connection between the resolution of singularities and Lie algebras. For instance, taking $m = n$ and identifying the $SU(m)$ gauge symmetry with $SU(n)$ by equating the V_1 and V_2 volumes, which imply in turn that $g_1 = g_2$, the bi-fundamental matter becomes then an adjoint one. This property is more transparent in the language of the representation theory. The adjoint of $SU(n + m)$ splits into $SU(n) \times SU(m)$ representations as:

$$(n + m)\overline{(n + m)} = n.\bar{n} + m.\bar{m} + \bar{n}.m + n.\bar{m}, \qquad (3.2)$$

where $n.\bar{n} + m.\bar{m}$ gives the gauge fields and $\bar{n}.m + n.\bar{m}$ define the bi-fundamental matters.

5

We note that $N = 2$ four-dimensional QFT's are represented by quiver diagrams where to each SU gauge group factor one associates a node and for each pair of groups with bi-fundamental matter, the two corresponding nodes are connected with a line. These diagrams are similar to the ADE Dynkin graphs.

4 $N = 1$ ADE quiver models from M-theory compactification

In this talk we discuss a straightforward way of elevating the geometric engineering of $N = 2$ gauge models in type IIA superstring theory to a similar construction of $N = 1$ ADE quiver gauge models in M-theory [24][3]. The method we will be using here is quite similar to the geometric engineering of $N = 2$ QFT$_4$. Indeed, we start with a local description of M-theory compactification on a G_2 manifold with K3 fibration over a real three-dimensional base space B_3. The scenario of type IIA superstring theory in six dimensions appears in seven dimensions in M-theory. In this way, the type IIA D2-branes are replaced by M2-branes, and we end with a seven-dimensional pure gauge models.

To get models with only four supercharges in four dimensions, we need to compactify the seven-dimensional model on a three-dimensional base preserving $1/4$ of the remaining 16 supercharges. The internal space must have vanishing first Betti number, $b_1 = 0$, to meet the requirement of G_2 holonomy. An example of such a geometry has the three-dimensional sphere $\mathbf{S^3}$ as base. However, in this case we obtain only *pure $N = 1$* Yang-Mills theory. The incorporation of matter may be achieved by introducing a non-trivial geometry in the base of the K3 fibration. This leads us to consider a three-dimensional intersecting geometry to describe a product gauge group with bi-fundamental matter in four dimensions.

Our method here is quite simple and motivated by the work on Lagrangian sub-manifolds in Calabi-Yau manifolds [27]. To do so, we consider the three-dimensional base space as a two-dimensional fibration over a one-dimensional base, where the fiber and the base each preserves half of the seven-dimensional M-theory supercharges. The entire base space B_3 could be embedded in a three-dimensional complex Calabi-Yau threefolds. The latter are realized explicitly as a family of (deformed) ADE singular K3 surfaces over the complex plane. In this way our base geometry can be identified

---

[3]Alternative studies of four-dimensional $N = 1$ gauge models in the framework of F- or M-theory may be found in [25, 26].

with Lagrangian sub-manifolds in such Calabi-Yau manifolds. It is easy to see that non-trivial three-cycles, satisfying the constraints of the G_2 base geometry, constitute ADE intersecting two-cycles of K3 surfaces fibered over a line segment in the complex plane.

For simplicity, let us consider the case of A_1 singularity. This is subsequently extended to more general ADE geometries. The deformed A_1 geometry is given by

$$z_1^2 + z_2^2 + z_3^2 = \mu \qquad (4.1)$$

where μ is a complex parameter. The A_1 threefolds may be obtained by varying the parameter μ over the complex plane parametrized by w

$$z_1^2 + z_2^2 + z_3^2 = \mu(w). \qquad (4.2)$$

The base space B_3 can then be viewed as a finite line segment with an $\mathbf{S^2}$ fibration, where the radius r of $\mathbf{S^2}$ vanishes at the two interval end points, and at the end points only [10]. The latter requirement ensures that no unwanted singularities are introduced. One way to realize this geometry is

$$r \sim \sin x \qquad (4.3)$$

where x is a real variable parameterizing the interval $[0, \pi]$ in the w-plane.

This construction may be extended to more complicated geometries where we have intersecting spheres according to affine ADE Dynkin diagrams. This extension has a nice toric geometry realization, where the $\mathbf{S^2}$ can be viewed as a segment $[v_1, v_2]$ with a circle on top, where it shrinks at the end points v_1 and v_2 [28]. On the other hand, $\mathbf{S^3}$ can be viewed as an interval with two circles on top, where a $\mathbf{S^1}$ shrinks at the first end and the other $\mathbf{S^1}$ shrinks at the second end. In the resolved elliptic singularity, $\mathbf{S^1} \times \mathbf{S^1}$ can be identified with a collection of two-cycles B_2 according to affine ADE Dynkin diagrams. In this way, the intersecting matrix of this geometry is given by the Cartan matrix of ADE affine Lie algebra.

Having determined the base geometry of the G_2 manifold with K3 fibration, we will discuss the corresponding gauge theory of the compactified M-theory using a type IIA superstring dual description. Indeed, M-theory on G_2 manifolds can be related to type IIA superstrings on Calabi-Yau threefolds in the presence of D6-branes wrapping Lagrangian sub-manifolds and filling the four-dimensional Minkowski space [9]. For instance, a local description of M-theory near the A_{n-1} singularity of K3 surfaces is equivalent to n units of D6-branes [28]. Indeed, on the seven-dimensional world-volume of each D6-brane we have a $U(1)$ symmetry. When the n D6-branes approach each other, the gauge symmetry is enhanced from $U(1)^{\otimes n}$ to $U(n)$. An extra

7

compactification of M-theory down to four-dimensional space-time is equivalent to wrapping D6-branes on the same geometry. In this way, the D6-brane physics can be achieved by using the method of (p, q) webs used in type II superstrings on toric Calabi-Yau manifolds [10, 11, 12, 13, 14]. Using the results of this method, we expect that the gauge model in four-dimensional Minkowski space has gauge group

$$G = \bigotimes_i U(n_i). \tag{4.4}$$

The integers n_i are specified by the anomaly cancellation condition. This means that they should form a null vector of the intersection matrix of the three-cycles I_{ij}. In the infra-red limit the $U(1)$ factors decouple and one is left with the gauge group $G = \bigotimes_{i=1}^m SU(n_i)$. The gauge group and matter content depend on the intersecting geometry in the three-dimensional base of the G_2 manifold. Identifying the base with a collection of the three-cycles being $\mathbf{S^2}$ fibrations over a line segment, the intersection matrix of the three-cycles can be identified with the Cartan matrix, K, of the associated ADE Lie algebra: $I_{ij} = -K_{ij}$. The anomaly cancellation condition is now translated into a condition on the affine Lie algebra, so the gauge group becomes $G = \bigotimes_i SU(s_i n)$. where s_i are the corresponding Dynkin numbers. The resulting models are $N = 1$ four-dimensional quiver models with bifundamental matter. They are represented by ADE affine Dynkin diagrams.

In this talk we have discussed the geometric engineering of $N = 1$ four-dimensional quiver models. In particular, we have considered models embedded in M-theory on a G_2 manifold with K3 fibration over a three-dimensional base space with ADE geometry. This base geometry is identified with ADE intersecting two-cycles over a line segment. Using the connection between M-theory and D6-brane physics of type IIA superstring, we have given the physics content of M-theory compactified on such a G_2 manifold. The corresponding gauge model has been discussed in terms of (p, q) brane webs.

References

[1] S. Kachru, C. Vafa, *Exact results for $N = 2$ compactifications of heterotic strings*, Nucl. Phys. **B450** (1995) 69, hep-th/9505105.

[2] A. Klemm, W. Lerche, P. Mayr, C. Vafa, N. Warner, *Self-dual strings and $N = 2$ supersymmetric field theory*, Nucl. Phys. **B477** (1996) 746, hep-th/9604034.

[3] S. Katz, C. Vafa, *Matter from geometry*, Nucl. Phys. **B497** (1997) 146, hep-th/9606086.

[4] S. Katz, A. Klemm, C. Vafa, *Geometric engineering of quantum field theories*, Nucl. Phys. **B497** (1997) 173, hep-th/9609239.

[5] M. Bershadsky, A. Johansen, T. Pantev, V. Sadov, C. Vafa, *F-theory, geometric engineering and $N = 1$ dualities*, Nucl. Phys. **B505** (1997) 153, hep-th/9612052.

[6] S. Katz, P. Mayr, C. Vafa, *Mirror symmetry and exact solution of $4d$ $N = 2$ gauge theories I*, Adv. Theor. Math. Phys. **1** (1998) 53, hep-th/9706110.

[7] A. Belhaj, A.E. Fallah, E.H. Saidi, *On the non-simply mirror geometries in type II strings*, Class. Quant. Grav. **17** (2000) 515.

[8] M. AitBenhaddou, A. Belhaj, E.H. Saidi, *Geometric Engineering of N=2 CFT$_4$s based on Indefinite Singularities: Hyperbolic Case*, hep-th/0307244. *Classification of N=2 supersymmetric CFT$_4$s: Indefinite Series*, hep-th/0308005.

[9] M. Cvetic, G. Shiu, A.M. Uranga, *Three-family supersymmetric standard-like models from intersecting brane worlds*, Phys. Rev. Lett. **87** (2001) 201801, hep-th/0107143; *Chiral four-dimensional $N = 1$ supersymmetric type IIA orientifolds from intersecting D6-branes*, Nucl. Phys. **B615** (2001) 3, hep-th/0107166; *Chiral type II orientifold constructions as M theory on G_2 holonomy spaces*, hep-th/0111179.

[10] A. Hanany, A. Iqbal, *Quiver theories from D6-branes via mirror symmetry*, JHEP **0204** (2002) 009, hep-th/010813.

[11] B. Feng, A. Hanany, Y.-H. He, A. Iqbal, *Quiver theories, soliton spectra and Picard-Lefschetz transformations*, hep-th/0206152.

[12] S. Franco, A. Hanany, *Geometric dualities in 4d field theories and their 5d interpretation*, hep-th/0207006; *Toric duality, Seiberg duality and Picard-Lefschetz transformations*, hep-th/0212299.

[13] A.M. Uranga, *Chiral four-dimensional string compactifications with intersecting D-branes*, hep-th/0301032.

[14] D. Cremades, L.E. Ibanez, F. Marchesano, *Yukawa couplings in intersecting D-brane models*, hep-th/0302105.

[15] C. Vafa, *Superstrings and Topological Strings at Large N*, J.Math.Phys. **42**(2001) 2798, hep-th/0008142.

[16] M. Atiyah, J. Maldacena, C. Vafa, *An M-theory Flop as a Large N Duality* J.Math.Phys. **42** (2001) 3209, hep-th/0011256 .

[17] S. Sinha, C. Vafa, *SO and Sp Chern-Simons at Large N*, hep-th/0012136.

[18] F. Cachazo, K. Intriligator, C. Vafa, *A Large N Duality via a Geometric Transition*, Nucl.Phys. **B603** (2001) 3, hep-th/0103067.

[19] F. Cachazo, S. Katz, C. Vafa, *Geometric Transitions and N=1 Quiver Theories*, hep-th/0108120 .

[20] F. Cachazo, B. Fiol, K. Intriligator, S. Katz, C. Vafa, *A Geometric Unification of Dualities*, Nucl.Phys. **B628** (2002) 3, hep-th/0110028.

[21] M. Aganagic, M. Marino, C. Vafa, *All Loop Topological String Amplitudes From Chern-Simons Theory*, hep-th/0206164 .

[22] Y.-H. He, G_2 *quivers*, JHEP **0302** (2003) 023, hep-th/0210127.

[23] A. Belhaj, *Comments on M-theory on G_2 manifolds and (p,q) webs*, hep-th/0303198.

[24] A. Belhaj, L.B. Drissi, J. Rasmussen, *On N=1 gauge models from geometric engineering in M-theory*, Class. Quantum Grav. **20** (2003) 4973, hep-th/0304019.

[25] C. Vafa, *On $N = 1$ Yang-Mills in four dimensions*, Adv. Theor. Math. Phys. **2** (1998) 497, hep-th/9801139.

[26] B.S. Acharya, *M theory, Joyce orbifolds and super Yang-Mills*, Adv. Theor. Math. Phys. **3** (1999) 227, hep-th/9812205.

[27] S. Gukov, C. Vafa, E. Witten, *CFT's from Calabi-Yau four-folds*, Nucl. Phys. **B584** (2000) 69, Erratum-ibid. **B608** (2001) 477, hep-th/9906070.

[28] N.C. Leung, C. Vafa, *Branes and toric geometry*, Adv. Theor. Math. Phys. **2** (1998) 91, hep-th/9711013.